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FUNDAMENTALS OF Differential Equations

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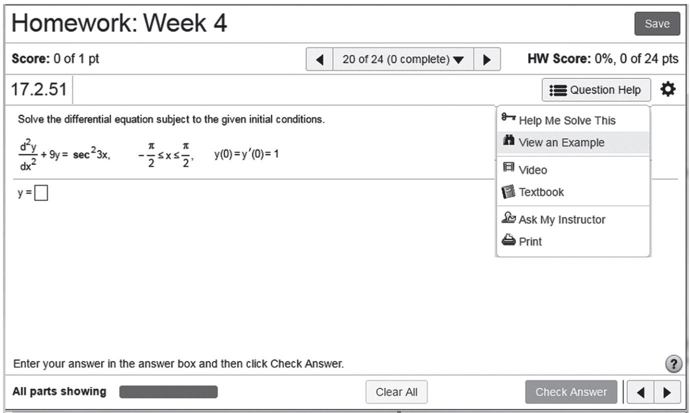

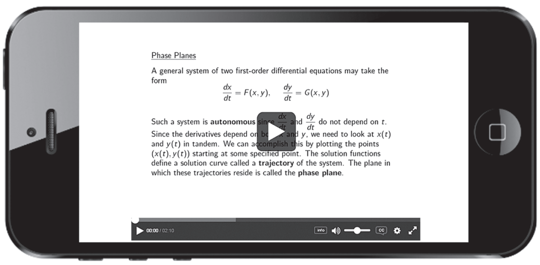

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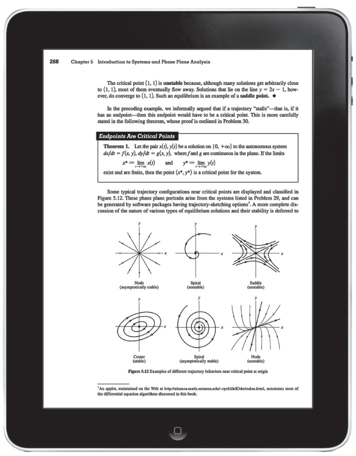

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NINTH EDITION

Fundamentals of Differential Equations

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Pearson

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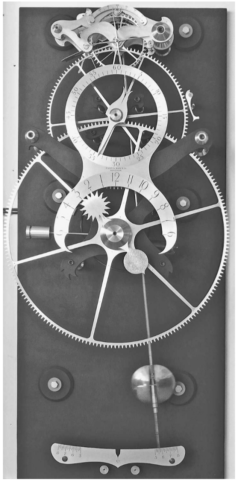

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Dedicated to R. Kent Nagle

He has left his imprint not only on these pages but upon all who knew him. He was that rare mathematician who could effectively communicate at all levels, imparting his love for the subject with the same ease to undergraduates, graduates, precollege students, public school teachers, and his colleagues at the University of South Florida.

Kent was at peace in life—a peace that emanated from the depth of his understanding of the human condition and the strength of his beliefs in the institutions of family, religion, and education. He was a research mathematician, an accomplished author, a Sunday school teacher, and a devoted husband and father.

Kent was also my dear friend and my jogging partner who has left me behind still struggling to keep pace with his high ideals.

E. B. Saff

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Contents

Preface

Our Goal

Fundamentals of Differential Equations is designed to serve the needs of a one-semester course in basic theory as well as applications of differential equations. The flexibility of the text provides the instructor substantial latitude in designing a syllabus to match the emphasis of the course. Sample syllabi are provided in this preface that illustrate the inherent flexibility of this text to balance theory, methodology, applications, and numerical methods, as well as the incorporation of commercially available computer software for this course.

New to This Edition

- This text now features a MyMathLab course with approximately 750 algorithmic online homework exercises, tutorial videos, and the complete eText. Please see the "Technology and Supplements" section below for more details.
- In the Laplace Transforms chapter (7), the treatments of discontinuous and periodic functions are now divided into two sections that are more appropriate for 50 minute lectures: Section 7.6 "Transforms of Discontinuous Functions" (page 383) and Section 7.7 "Transforms of Periodic and Power Functions" (page 392).
- New examples have been added dealing with variation of parameters, Laplace transforms, the Gamma function, and eigenvectors (among others).
- New problems added to exercise sets deal with such topics as axon gating variables and oscillations of a helium-filled balloon on a cord. Additionally, novel problems accompany the new projects, focusing on economic models, disease control, synchronization, signal propagation, and phase plane analyses of neural responses. We have also added a set of Review Problems for Chapter 1 (page 29).
- Several pedagogical changes were made including amplification of the distinction between phase plane solutions and actual trajectories in Chapter 5 and incorporation of matrix and Jacobian formulations for autonomous systems.
- A new appendix lists commercial software and freeware for direction fields, phase portraits, and numerical methods for solving differential equations. (Appendix G, page A-17.)
- "The 2014–2015 Ebola Epidemic" is a new Project in Chapter 5 that describes a system of differential equations for modelling for the spread of the disease in West Africa. The model incorporates such features as contact tracing, number of contacts, likelihood of infection, and efficacy of isolation. See Project F, page 314.
- A new project in Chapter 1 called "Applications to Economics" deals with models for an agrarian economy as well as the growth of capital. See Project C, page 35.

- A new project in Chapter 4 called "Gravity Train" invites to reader to utilize differential equations in the design of an underground tunnel from Moscow to St. Petersburg, Russia, using gravity for propulsion. See Project H, page 240.
- Phase-locked loops constitute the theme of a new project in Chapter 5 that utilizes differential equations to analyze a technique for measuring or matching high frequency radio oscillations. See Project G, page 317.
- A new Project in Chapter 10 broadens the analysis of the wave and heat equations to explore the telegrapher's and cable equations. See Project E, page 637.

Prerequisites

While some universities make linear algebra a prerequisite for differential equations, many schools (especially engineering) only require calculus. With this in mind, we have designed the text so that only Chapter 6 (Theory of Higher-Order Linear Differential Equations) and Chapter 9 (Matrix Methods for Linear Systems) require more than high school level linear algebra. Moreover, Chapter 9 contains review sections on matrices and vectors as well as specific references for the deeper results used from the theory of linear algebra. We have also written Chapter 5 so as to give an introduction to systems of differential equations—including methods of solving, phase plane analysis, applications, numerical procedures, and Poincaré maps—that does not require a background in linear algebra.

Sample Syllabi

As a rough guide in designing a one-semester syllabus related to this text, we provide three samples that can be used for a 15-week course that meets three hours per week. The first emphasizes applications and computations including phase plane analysis; the second is designed for courses that place more emphasis on theory; and the third stresses methodology and partial differential equations. Chapters 1, 2, and 4 provide the core for any first course. The rest of the chapters are, for the most part, independent of each other. For students with a background in linear algebra, the instructor may prefer to replace Chapter 7 (Laplace Transforms) or Chapter 8 (Series Solutions of Differential Equations) with sections from Chapter 9 (Matrix Methods for Linear Systems).

	Methods, Computations, and Applications	Theory and Methods (linear algebra prerequisite)	Methods and Partial Differential Equations
Week	**Sections**	**Sections**	**Sections**
1	1.1, 1.2, 1.3	1.1, 1.2, 1.3	1.1, 1.2, 1.3
2	1.4, 2.2	1.4, 2.2, 2.3	1.4, 2.2
3	2.3, 2.4, 3.2	2.4, 3.2, 4.1	2.3, 2.4
4	3.4, 3.5, 3.6	4.2, 4.3, 4.4	3.2, 3.4
5	3.7, 4.1	4.5, 4.6	4.2, 4.3
6	4.2, 4.3, 4.4	4.7, 5.2, 5.3	4.4, 4.5, 4.6
7	4.5, 4.6, 4.7	5.4, 6.1	4.7, 5.1, 5.2
8	4.8, 4.9	6.2, 6.3, 7.2	7.1, 7.2, 7.3
9	4.10, 5.1, 5.2	7.3, 7.4, 7.5	7.4, 7.5
10	5.3, 5.4, 5.5	7.6, 7.7, 7.8	7.6, 7.7
11	5.6, 5.7, 7.2	8.2, 8.3	7.8, 8.2
12	7.3, 7.4, 7.5	8.4, 8.6, 9.1	8.3, 8.5, 8.6
13	7.6, 7.7, 7.8	9.2, 9.3	10.2, 10.3
14	8.1, 8.2, 8.3	9.4, 9.5, 9.6	10.4, 10.5
15	8.4, 8.6	9.7, 9.8	10.6, 10.7

Retained Features

Flexible Organization Most of the material is modular in nature to allow for various course configurations and emphasis (theory, applications and techniques, and concepts).

Optional Use of Computer Software The availability of computer packages such as Mathcad®, Mathematica®, MATLAB®, and Maple™ provides an opportunity for the student to conduct numerical experiments and tackle realistic applications that give additional insights into the subject. Consequently, we have inserted several exercises and projects throughout the text that are designed for the student to employ available software in phase plane analysis, eigenvalue computations, and the numerical solutions of various equations.

Review of Integration In response to the perception that many of today's students' skills in integration have gotten rusty by the time they enter a differential equations course, we have included an appendix offering a quick review of the basic methods for integrating functions analytically.

Choice of Applications Because of syllabus constraints, some courses will have little or no time for sections (such as those in Chapters 3 and 5) that exclusively deal with applications. Therefore, we have made the sections in these chapters independent of each other. To afford the instructor even greater flexibility, we have built in a variety of applications in the exercises for the theoretical sections. In addition, we have included many projects that deal with such applications.

Projects At the end of each chapter are projects relating to the material covered in the chapter. Several of them have been contributed by distinguished researchers. A project might involve a more challenging application, delve deeper into the theory, or introduce more advanced topics in differential equations. Although these projects can be tackled by an individual student, classroom testing has shown that working in groups lends a valuable added dimension to the learning experience. Indeed, it simulates the interactions that take place in the professional arena.

Technical Writing Exercises Communication skills are, of course, an essential aspect of professional activities. Yet few texts provide opportunities for the reader to develop these skills. Thus, we have added at the end of most chapters a set of clearly marked technical writing exercises that invite students to make documented responses to questions dealing with the concepts in the chapter. In so doing, students are encouraged to make comparisons between various methods and to present examples that support their analysis.

Historical Footnotes Throughout the text historical footnotes are set off by colored daggers (†). These footnotes typically provide the name of the person who developed the technique, the date, and the context of the original research.

Motivating Problem Most chapters begin with a discussion of a problem from physics or engineering that motivates the topic presented and illustrates the methodology.

Chapter Summary and Review Problems All of the main chapters contain a set of review problems along with a synopsis of the major concepts presented.

Computer Graphics Most of the figures in the text were generated via computer. Computer graphics not only ensure greater accuracy in the illustrations, they demonstrate the use of numerical experimentation in studying the behavior of solutions.

Proofs While more pragmatic students may balk at proofs, most instructors regard these justifications as an essential ingredient in a textbook on differential equations. As with any text at this level, certain details in the proofs must be omitted. When this occurs, we flag the instance and refer readers either to a problem in the exercises or to another text. For convenience, the end of a proof is marked by the symbol ◆.

Linear Theory We have developed the theory of linear differential equations in a gradual manner. In Chapter 4 (Linear Second-Order Equations) we first present the basic theory for linear second-order equations with constant coefficients and discuss various techniques for solving these equations. Section 4.7 surveys the extension of these ideas to variable-coefficient second-order equations. A more general and detailed discussion of linear differential equations is given in Chapter 6 (Theory of Higher-Order Linear Differential Equations). For a beginning course emphasizing methods of solution, the presentation in Chapter 4 may be sufficient and Chapter 6 can be skipped.

Numerical Algorithms Several numerical methods for approximating solutions to differential equations are presented along with program outlines that are easily implemented on a computer. These methods are introduced early in the text so that teachers and/or students can use them for numerical experimentation and for tackling complicated applications. Where appropriate we direct the student to software packages or web-based applets for implementation of these algorithms.

Exercises An abundance of exercises is graduated in difficulty from straightforward, routine problems to more challenging ones. Deeper theoretical questions, along with applications, usually occur toward the end of the exercise sets. Throughout the text we have included problems and projects that require the use of a calculator or computer. These exercises are denoted by the symbol .

Laplace Transforms We provide a detailed chapter on Laplace transforms (Chapter 7), since this is a recurring topic for engineers. Our treatment emphasizes discontinuous forcing terms and includes a section on the Dirac delta function.

Power Series Power series solutions is a topic that occasionally causes student anxiety. Possibly, this is due to inadequate preparation in calculus where the more subtle subject of convergent series is (frequently) covered at a rapid pace. Our solution has been to provide a graceful initiation into the theory of power series solutions with an exposition of Taylor polynomial approximants to solutions, deferring the sophisticated issues of convergence to later sections. Unlike many texts, ours provides an extensive section on the method of Frobenius (Section 8.6) as well as a section on finding a second linearly independent solution. While we have given considerable space to power series solutions, we have also taken great care to accommodate the instructor who only wishes to give a basic introduction to the topic. An introduction to solving differential equations using power series and the method of Frobenius can be accomplished by covering the materials in Sections 8.1, 8.2, 8.3, and 8.6.

Partial Differential Equations An introduction to this subject is provided in Chapter 10, which covers the method of separation of variables, Fourier series, the heat equation, the wave equation, and Laplace's equation. Examples in two and three dimensions are included.

Phase Plane Chapter 5 describes how qualitative information for two-dimensional systems can be gleaned about the solutions to intractable autonomous equations by observing their direction fields and critical points on the phase plane. With the assistance of suitable software, this approach provides a refreshing, almost recreational alternative to the traditional analytic methodology as we discuss applications in nonlinear mechanics, ecosystems, and epidemiology.

Vibrations Motivation for Chapter 4 on linear differential equations is provided in an introductory section describing the mass–spring oscillator. We exploit the reader's familiarity with common vibratory motions to anticipate the exposition of the theoretical and analytical aspects of linear equations. Not only does this model provide an anchor for the discourse on constant-coefficient equations, but a liberal interpretation of its features enables us to predict the qualitative behavior of variable-coefficient and nonlinear equations as well.

Review of Algebraic Equations and Matrices The chapter on matrix methods for linear systems (Chapter 9) begins with two (optional) introductory sections reviewing the theory of linear algebraic systems and matrix algebra.

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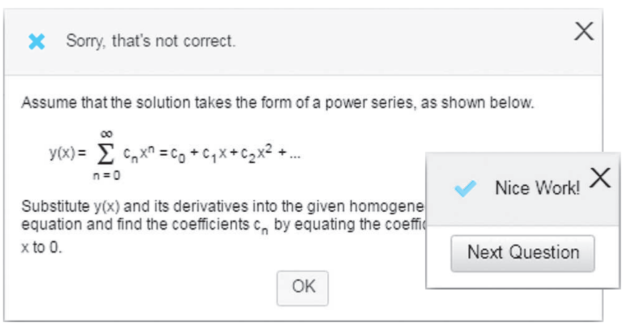

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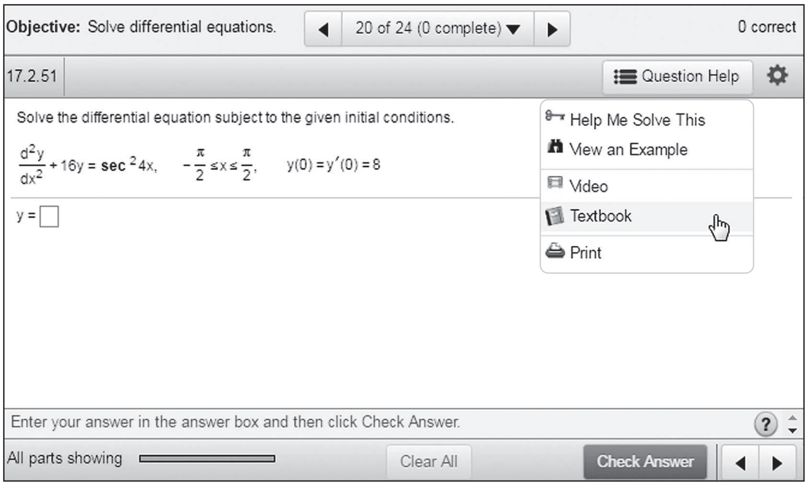

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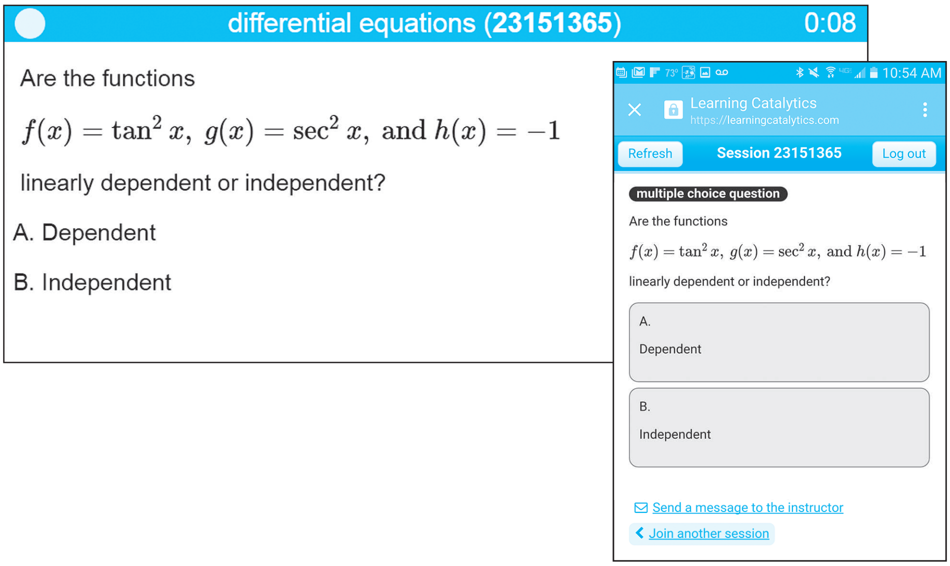

- **Instructional videos** are available as learning aids within exercises and for self-study within the Multimedia Library. The Guide to Video-Based Assignments makes it easy to assign videos for homework by showing which MyMathLab exercises correspond to each video.
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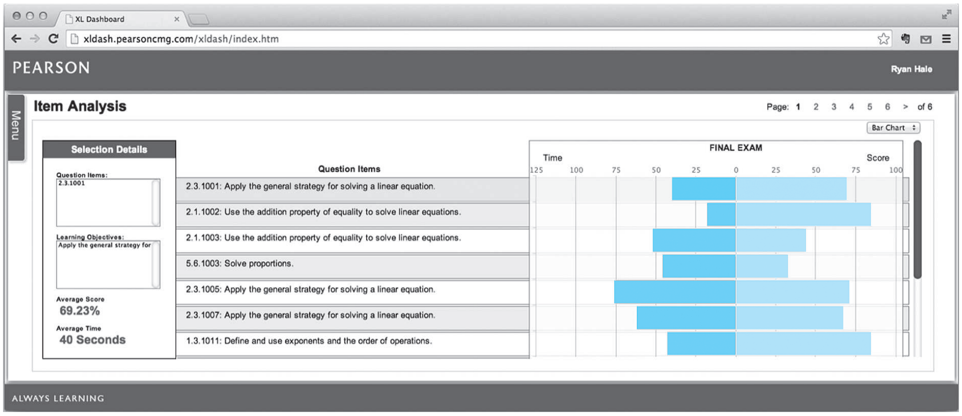

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By Thomas W. Polaski (Winthrop University), Bruno Welfert (Arizona State University), and Maurino Bautista (Rochester Institute of Technology), respectively. These manuals contain a collection of instructor tips, worksheets, and projects to aid instructors in integrating computer algebra systems into their courses. Complete manuals are available for instructor download within MyMathLab. Student worksheets and projects available for download within MyMathLab.

Acknowledgments

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CHAPTER

1 Introduction

1.1 Background

In the sciences and engineering, mathematical models are developed to aid in the understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a **differential equation.** Two examples of models developed in calculus are the free fall of a body and the decay of a radioactive substance.

In the case of free fall, an object is released from a certain height above the ground and falls under the force of gravity.† Newton's second law, which states that an object's mass times its acceleration equals the total force acting on it, can be applied to the falling object. This leads to the equation (see Figure 1.1)

$$m\frac{d^2h}{dt^2} = -mg\ ,$$

where m is the mass of the object, h is the height above the ground, d^2h/dt^2 is its acceleration, g is the (constant) gravitational acceleration, and $-mg$ is the force due to gravity. This is a differential equation containing the second derivative of the unknown height h as a function of time.

Fortunately, the above equation is easy to solve for h. All we have to do is divide by m and integrate twice with respect to t. That is,

$$\frac{d^2h}{dt^2} = -g\ ,$$

so

$$\frac{dh}{dt} = -gt + c_1$$

and

$$h = h(t) = \frac{-gt^2}{2} + c_1t + c_2\ .$$

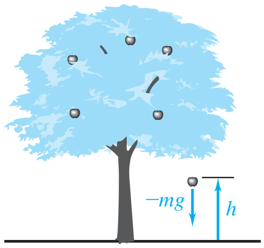

Figure 1.1 Apple in free fall

†We are assuming here that gravity is the *only* force acting on the object and that this force is constant. More general models would take into account other forces, such as air resistance.

We will see that the constants of integration, c_1 and c_2, are determined if we know the *initial* height and the *initial* velocity of the object. We then have a formula for the height of the object at time t.

In the case of radioactive decay (Figure 1.2), we begin from the premise that the rate of decay is proportional to the amount of radioactive substance present. This leads to the equation

$$\frac{dA}{dt} = -kA\,, \qquad k > 0\,,$$

where $A(>0)$ is the unknown amount of radioactive substance present at time t and k is the proportionality constant. To solve this differential equation, we rewrite it in the form

$$\frac{1}{A}dA = -k\,dt$$

and integrate to obtain

$$\int \frac{1}{A}dA = \int -k\,dt$$

$$\ln A + C_1 = -kt + C_2\,.$$

Solving for A yields

$$A = A(t) = e^{\ln A} = e^{-kt}e^{C_2 - C_1} = Ce^{-kt},$$

where C is the combination of integration constants $e^{C_2 - C_1}$. The value of C, as we will see later, is determined if the *initial* amount of radioactive substance is given. We then have a formula for the amount of radioactive substance at any future time t.

Even though the above examples were easily solved by methods learned in calculus, they do give us some insight into the study of differential equations in general. First, notice that the solution of a differential equation is a *function,* like $h(t)$ or $A(t)$, not merely a number. Second, integration† is an important tool in solving differential equations (not surprisingly!). Third, we cannot expect to get a unique solution to a differential equation, since there will be arbitrary "constants of integration." The second derivative d^2h/dt^2 in the free-fall equation gave rise to two constants, c_1 and c_2, and the first derivative in the decay equation gave rise, ultimately, to one constant, C.

Whenever a mathematical model involves the **rate of change** of one variable with respect to another, a differential equation is apt to appear. Unfortunately, in contrast to the examples for free fall and radioactive decay, the differential equation may be very complicated and difficult to analyze.

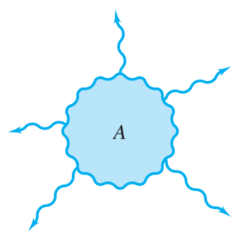

Figure 1.2 Radioactive decay

†For a review of integration techniques, see Appendix A.

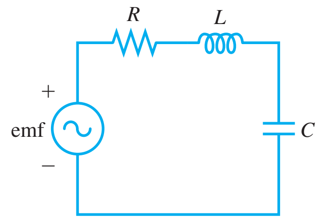

Figure 1.3 Schematic for a series RLC circuit

Differential equations arise in a variety of subject areas, including not only the physical sciences but also such diverse fields as economics, medicine, psychology, and operations research. We now list a few specific examples.

1. In banking practice, if $P(t)$ is the number of dollars in a savings bank account that pays a yearly interest rate of $r\%$ *compounded continuously*, then P satisfies the differential equation

(1) $$\frac{dP}{dt} = \frac{r}{100}P\,, \quad t \text{ in years.}$$

2. A classic application of differential equations is found in the study of an electric circuit consisting of a resistor, an inductor, and a capacitor driven by an electromotive force (see Figure 1.3). Here an application of Kirchhoff's laws† leads to the equation

(2) $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)\,,$$

where L is the inductance, R is the resistance, C is the capacitance, $E(t)$ is the electromotive force, $q(t)$ is the charge on the capacitor, and t is the time.

3. In psychology, one model of the learning of a task involves the equation

(3) $$\frac{dy/dt}{y^{3/2}(1-y)^{3/2}} = \frac{2p}{\sqrt{n}}\,.$$

Here the variable y represents the learner's skill level as a function of time t. The constants p and n depend on the individual learner and the nature of the task.

4. In the study of vibrating strings and the propagation of waves, we find the *partial* differential equation

(4) $$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0\,,^{\ddagger}$$

where t represents time, x the location along the string, c the wave speed, and u the displacement of the string, *which is a function of time and location*.

†We will discuss Kirchhoff's laws in Section 3.5.

‡*Historical Footnote:* This partial differential equation was first discovered by Jean le Rond d'Alembert (1717–1783) in 1747.

To begin our study of differential equations, we need some common terminology. If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter an **independent variable.** Thus, in the equation

(5) $$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0\,,$$

t is the independent variable and x is the dependent variable. We refer to a and k as **coefficients** in equation (5). In the equation

(6) $$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y\,,$$

x and y are independent variables and u is the dependent variable.

A differential equation involving only ordinary derivatives with respect to a single independent variable is called an **ordinary differential equation.** A differential equation involving partial derivatives with respect to more than one independent variable is a **partial differential equation.** Equation (5) is an ordinary differential equation, and equation (6) is a partial differential equation.

The **order** of a differential equation is the order of the highest-order derivatives present in the equation. Equation (5) is a second-order equation because d^2x/dt^2 is the highest-order derivative present. Equation (6) is a first-order equation because only first-order partial derivatives occur.

It will be useful to classify ordinary differential equations as being either linear or nonlinear. Remember that lines (in two dimensions) and planes (in three dimensions) are especially easy to visualize, when compared to nonlinear objects such as cubic curves or quadric surfaces. For example, all the points on a line can be found if we know just two of them. Correspondingly, *linear* differential equations are more amenable to solution than nonlinear ones. Observe that the equations for lines $ax + by = c$ and planes $ax + by + cz = d$ have the feature that the variables appear in *additive combinations of their first powers only*. By analogy a **linear differential equation** is one in which the dependent variable y and its derivatives appear in additive combinations of their first powers.

More precisely, a differential equation is **linear** if it has the format

(7) $$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)\,,$$

where $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $F(x)$ depend only on the independent variable x. The additive combinations are permitted to have multipliers (coefficients) that depend on x; no restrictions are made on the nature of this x-dependence. If an ordinary differential equation is not linear, then we call it **nonlinear.** For example,

$$\frac{d^2y}{dx^2} + y^3 = 0$$

is a nonlinear second-order ordinary differential equation because of the y^3 term, whereas

$$t^3\frac{dx}{dt} = t^3 + x$$

is linear (despite the t^3 terms). The equation

$$\frac{d^2y}{dx^2} - y\frac{dy}{dx} = \cos x$$

is nonlinear because of the $y\,dy/dx$ term.

Although the majority of equations one is likely to encounter in practice fall into the *nonlinear* category, knowing how to deal with the simpler linear equations is an important first step (just as tangent lines help our understanding of complicated curves by providing local approximations).

1.1 EXERCISES

In Problems 1–12, a differential equation is given along with the field or problem area in which it arises. Classify each as an ordinary differential equation (ODE) or a partial differential equation (PDE), give the order, and indicate the independent and dependent variables. If the equation is an ordinary differential equation, indicate whether the equation is linear or nonlinear.

1. $5\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = 2\cos 3t$

(mechanical vibrations, electrical circuits, seismology)

2. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

(Hermite's equation, quantum-mechanical harmonic oscillator)

3. $\frac{dy}{dx} = \frac{y(2-3x)}{x(1-3y)}$

(competition between two species, ecology)

4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(Laplace's equation, potential theory, electricity, heat, aerodynamics)

5. $y\left[1 + \left(\frac{dy}{dx}\right)^2\right] = C$, where C is a constant

(brachistochrone problem,† calculus of variations)

6. $\frac{dx}{dt} = k(4-x)(1-x)$, where k is a constant

(chemical reaction rates)

7. $\frac{dp}{dt} = kp(P-p)$, where k and P are constants

(logistic curve, epidemiology, economics)

8. $\sqrt{1-y}\,\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

(Kidder's equation, flow of gases through a porous medium)

9. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$

(aerodynamics, stress analysis)

10. $8\frac{d^4y}{dx^4} = x(1-x)$

(deflection of beams)

11. $\frac{\partial N}{\partial t} = \frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + kN$, where k is a constant

(nuclear fission)

12. $\frac{d^2y}{dx^2} - 0.1(1-y^2)\frac{dy}{dx} + 9y = 0$

(van der Pol's equation, triode vacuum tube)

In Problems 13–16, write a differential equation that fits the physical description.

13. The rate of change of the population p of bacteria at time t is proportional to the population at time t.

14. The velocity at time t of a particle moving along a straight line is proportional to the fourth power of its position x.

15. The rate of change in the temperature T of coffee at time t is proportional to the difference between the temperature M of the air at time t and the temperature of the coffee at time t.

16. The rate of change of the mass A of salt at time t is proportional to the square of the mass of salt present at time t.

17. **Drag Race.** Two drivers, Alison and Kevin, are participating in a drag race. Beginning from a standing start, they each proceed with a constant acceleration. Alison covers the last $1/4$ of the distance in 3 seconds, whereas Kevin covers the last $1/3$ of the distance in 4 seconds. Who wins and by how much time?

†*Historical Footnote:* In 1630 Galileo formulated the brachistochrone problem ($\beta\rho\acute{\alpha}\chi\acute{\iota}\sigma\tau o\varsigma$ = shortest, $\chi\rho\acute{o}\nu o\varsigma$ = time), that is, to determine a path down which a particle will fall from one given point to another in the shortest time. It was reproposed by John Bernoulli in 1696 and solved by him the following year.

1.2 Solutions and Initial Value Problems

An nth-order ordinary differential equation is an equality relating the independent variable to the nth derivative (and usually lower-order derivatives as well) of the dependent variable. Examples are

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x^3 \text{ (second-order, } x \text{ independent, } y \text{ dependent)}$$

$$\sqrt{1-\left(\frac{d^2y}{dt^2}\right)}-y=0 \text{ (second-order, } t \text{ independent, } y \text{ dependent)}$$

$$\frac{d^4x}{dt^4}=xt \text{ (fourth-order, } t \text{ independent, } x \text{ dependent).}$$

Thus, a general form for an nth-order equation with x independent, y dependent, can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^ny}{dx^n}\right)=0\,, \tag{1}$$

where F is a function that depends on x, y, and the derivatives of y up to order n; that is, on x, y, . . . , d^ny/dx^n. We assume that the equation holds for all x in an open interval I ($a<x<b$, where a or b could be infinite). In many cases we can isolate the highest-order term d^ny/dx^n and write equation (1) as

$$\frac{d^ny}{dx^n}=f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right), \tag{2}$$

which is often preferable to (1) for theoretical and computational purposes.

Explicit Solution

Definition 1. A function $\phi(x)$ that when substituted for y in equation (1) [or (2)] satisfies the equation for all x in the interval I is called an **explicit solution** to the equation on I.

Example 1 Show that $\phi(x)=x^2-x^{-1}$ is an explicit solution to the linear equation

$$\frac{d^2y}{dx^2}-\frac{2}{x^2}y=0\,, \tag{3}$$

but $\psi(x)=x^3$ is not.

Solution The functions $\phi(x)=x^2-x^{-1}$, $\phi'(x)=2x+x^{-2}$, and $\phi''(x)=2-2x^{-3}$ are defined for all $x\neq 0$. Substitution of $\phi(x)$ for y in equation (3) gives

$$(2-2x^{-3})-\frac{2}{x^2}(x^2-x^{-1})=(2-2x^{-3})-(2-2x^{-3})=0\,.$$

Since this is valid for any $x \neq 0$, the function $\phi(x) = x^2 - x^{-1}$ is an explicit solution to (3) on $(-\infty, 0)$ and also on $(0, \infty)$.

For $\psi(x) = x^3$ we have $\psi'(x) = 3x^2$, $\psi''(x) = 6x$, and substitution into (3) gives

$$6x - \frac{2}{x^2}x^3 = 4x = 0\,,$$

which is valid only at the point $x = 0$ and not on an interval. Hence $\psi(x)$ is not a solution. ◆

Example 2 Show that for *any* choice of the constants c_1 and c_2, the function

$$\phi(x) = c_1 e^{-x} + c_2 e^{2x}$$

is an explicit solution to the linear equation

$$y'' - y' - 2y = 0\,. \tag{4}$$

Solution We compute $\phi'(x) = -c_1 e^{-x} + 2c_2 e^{2x}$ and $\phi''(x) = c_1 e^{-x} + 4c_2 e^{2x}$. Substitution of ϕ, ϕ', and ϕ'' for y, y', and y'' in equation (4) yields

$$\begin{aligned}(c_1 e^{-x} + 4c_2 e^{2x}) - (-c_1 e^{-x} + 2c_2 e^{2x}) - 2(c_1 e^{-x} + c_2 e^{2x})\\ = (c_1 + c_1 - 2c_1)e^{-x} + (4c_2 - 2c_2 - 2c_2)e^{2x} = 0\,.\end{aligned}$$

Since equality holds for all x in $(-\infty, \infty)$, then $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ is an explicit solution to (4) on the interval $(-\infty, \infty)$ for any choice of the constants c_1 and c_2. ◆

As we will see in Chapter 2, the methods for solving differential equations do not always yield an explicit solution for the equation. We may have to settle for a solution that is defined implicitly. Consider the following example.

Example 3 Show that the relation

$$y^2 - x^3 + 8 = 0 \tag{5}$$

implicitly defines a solution to the nonlinear equation

$$\frac{dy}{dx} = \frac{3x^2}{2y} \tag{6}$$

on the interval $(2, \infty)$.

Solution When we solve (5) for y, we obtain $y = \pm\sqrt{x^3 - 8}$. Let's try $\phi(x) = \sqrt{x^3 - 8}$ to see if it is an explicit solution. Since $d\phi/dx = 3x^2/\left(2\sqrt{x^3 - 8}\right)$, both ϕ and $d\phi/dx$ are defined on $(2, \infty)$. Substituting them into (6) yields

$$\frac{3x^2}{2\sqrt{x^3 - 8}} = \frac{3x^2}{2\left(\sqrt{x^3 - 8}\right)}\,,$$

which is indeed valid for all x in $(2, \infty)$. [You can check that $\psi(x) = -\sqrt{x^3 - 8}$ is also an explicit solution to (6).] ◆

Implicit Solution

Definition 2. A relation $G(x, y) = 0$ is said to be an **implicit solution** to equation (1) on the interval I if it defines one or more explicit solutions on I.

Example 4 Show that

(7) $$x + y + e^{xy} = 0$$

is an implicit solution to the nonlinear equation

(8) $$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0 .$$

Solution First, we observe that we are unable to solve (7) directly for y in terms of x alone. However, for (7) to hold, we realize that any change in x requires a change in y, so we expect the relation (7) to define implicitly at least one function $y(x)$. This is difficult to show directly but can be rigorously verified using the **implicit function theorem**[†] of advanced calculus, which guarantees that such a function $y(x)$ exists that is also differentiable (see Problem 30).

Once we know that y is a differentiable function of x, we can use the technique of implicit differentiation. Indeed, from (7) we obtain on differentiating with respect to x and applying the product and chain rules,

$$\frac{d}{dx}(x + y + e^{xy}) = 1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0$$

or

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0 ,$$

which is identical to the differential equation (8). Thus, relation (7) is an implicit solution on some interval guaranteed by the implicit function theorem. ◆

Example 5 Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to

(9) $$y\frac{dy}{dx} - 4x = 0 .$$

Graph the solution curves for $C = 0, \pm 1, \pm 4$. (We call the collection of all such solutions a *one-parameter family of solutions*.)

Solution When we implicitly differentiate the equation $4x^2 - y^2 = C$ with respect to x, we find

$$8x - 2y\frac{dy}{dx} = 0 ,$$

[†]See *Vector Calculus*, 6th ed, by J. E. Marsden and A. J. Tromba (Freeman, San Francisco, 2013).

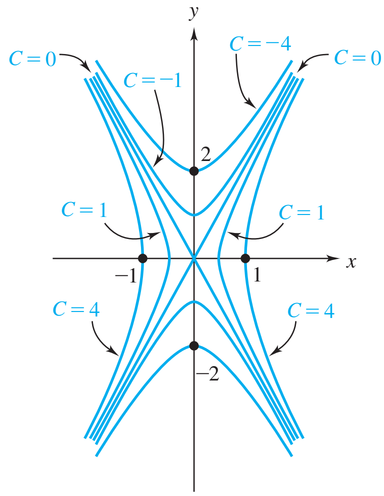

Figure 1.4 Implicit solutions $4x^2 - y^2 = C$

which is equivalent to (9). In Figure 1.4 we have sketched the implicit solutions for $C = 0, \pm 1, \pm 4$. The curves are hyperbolas with common asymptotes $y = \pm 2x$. Notice that the implicit solution curves (with C arbitrary) fill the entire plane and are nonintersecting for $C \neq 0$. For $C = 0$, the implicit solution gives rise to the two explicit solutions $y = 2x$ and $y = -2x$, both of which pass through the origin. ◆

For brevity we hereafter use the term *solution* to mean either an explicit or an implicit solution.

In the beginning of Section 1.1, we saw that the solution of the *second*-order free-fall equation invoked two arbitrary constants of integration c_1, c_2:

$$h(t) = \frac{-gt^2}{2} + c_1 t + c_2 ,$$

whereas the solution of the *first*-order radioactive decay equation contained a single constant C:

$$A(t) = Ce^{-kt} .$$

It is clear that integration of the simple *fourth*-order equation

$$\frac{d^4y}{dx^4} = 0$$

brings in four undetermined constants:

$$y(x) = c_1x^3 + c_2x^2 + c_3x + c_4 .$$

It will be shown later in the text that in general the methods for solving nth-order differential equations evoke n arbitrary constants. In most cases, we will be able to evaluate these constants if we know n initial values $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$.